Equations of elasticity

1.0 Hertz elastic contact stresses

$$\frac{1}{E^*} = \frac{1-u_1^2}{E_1} + \frac{1-u_2^2}{E_2}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

1.1 Line contacts (load per unit length f_n)

$$a = \left(\frac{4 f_n R}{pE^*}\right)^{1/2}$$

Maximum contact pressure

$$p_o = \frac{2 f_n}{pa} = \left(\frac{f_n E^*}{pR}\right)^{1/2}$$

Approach of distant points (ignoring end effects)

$$d = \frac{2 f_n}{p}\left\{\begin{aligned}&\frac{1-n_1^2}{E_1}\left(\ln\left(\frac{4R_1}{a}\right)-\frac{1}{2}\right)\\&+\frac{1-n_2^2}{E_2}\left(\ln\left(\frac{4R_2}{a}\right)-\frac{1}{2}\right)\end{aligned}\right\}$$

Maximum shear stress

$t_o = 0.30 p_o$ at $x = 0$, $z = 0.78a$

1.2 Circular contacts

$$a = \left(\frac{3RF_n}{4E^*}\right)^{1/3}$$

Maximum contact pressure

$$p_o = \frac{3F_n}{2pa^2} = \left(\frac{6E^{*2} F_n}{p^3 R^2}\right)^{1/3}$$

Approach of distant points

$$d = \frac{a^2}{R} = \left(\frac{9F_n^2}{16RE^{*2}}\right)^{1/3}$$

Maximum shear stress

$t_o = 0.31 p_o$ at $r = 0$, $z = 0.48a$

Maximum tensile stress

$s_o = \dfrac{(1-2u)p_o}{3}$ at $r = a$, $z = 0$

Pressure distribution in the contact circle

$$p = p_o\left(1-\left(\frac{r}{a}\right)^2\right)$$

1.2.1 Sliding contacts (coeff. of friction m)

Maximum tensile stress

$$s_t = \frac{3F_n}{2pa^2}\left(\frac{1-2u}{3} + \frac{4+u}{8}pm\right)$$

Von Mises stress at yield

$$\sqrt{2}s_{Yield} = \frac{p_o}{\sqrt{3}}\left\{\begin{aligned}&\frac{(1-2u)^2}{3} + \frac{(1-2u)(2-u)mp}{4}\\&\ldots-\frac{(16-4u+7u^2)m^2p^2}{64}\end{aligned}\right\}^{1/2}$$

2.0 Yield criteria for elastic/plastic materials

The von Mises criteria predicts yield at

$$(s_1 - s_2)^2 + (s_1 - s_3)^2 + (s_3 - s_2)^2 \geq 2s_{Yield}^2$$

for a simple direct stress plus shear, yield occurs when

$$s_{Yield}^2 \leq s_x^2 + 3t_{xy}^2$$

The Tresca criteria predicts yield at

$$\max.\left[|s_1 - s_2|,|s_1 - s_3|,|s_3 - s_2|\right] \geq s_{Yield}$$

s_1, s_2 & s_3 are principal stresses

for a simple direct stress plus shear yield occurs when

$$s_{Yield}^2 \leq s_x^2 + 4t_{xy}^2$$

3.0 Elastic bending

Bending about a principal axis

$$\frac{s}{y} = \frac{E}{R} = \frac{M}{I}$$

Second moment of area

$I = \dfrac{bd^3}{12}$ for a rectangular beam

$I = \dfrac{pD^4}{64}$ for a circular rod

4.0 St. Venant torsion of elastic members

4.1 Round shaft

$$\frac{T}{J} = \frac{Gq}{L} = \frac{t}{r}$$

where

$$J = \frac{pD^4}{32}$$

4.2 Flat rectangular strip

$$\frac{T}{J} = \frac{Gq}{L}$$

where

$$J = \frac{bt^3}{3}$$

5.0 Linear elasticity

$$G = \frac{E}{2(1+u)}$$

$$k = \frac{E}{3(1-2u)}$$

5.1 Uniaxial stress

$$e_1 = \frac{s_1}{E},\ e_2 = -ue_1 = e_3$$

similar expressions for e_2 & e_3

5.2 Biaxial stress

$$e_1 = \frac{s_1}{E} - u\frac{s_2}{E}$$

$$e_2 = \frac{s_2}{E} - u\frac{s_1}{E}$$

$$e_3 = -u\frac{s_2}{E} - u\frac{s_1}{E}$$

$$s_1 = -\frac{E(e_1 + ue_2)}{1 - u^2}$$

$$s_2 = -\frac{E(e_2 + ue_1)}{1 - u^2}$$

5.3 Triaxial stresses

$$e_1 = \frac{(s_1 - u(s_2 + s_3))}{E}$$

similar expressions for e_2 & e_3

$$s_1 = \frac{E(e_1(1-u) + u(e_2 + e_3))}{1 - u - 2u^2}$$

similar expressions for s_2 & s_3

6.0 Fatigue

6.1 Cycles to failure

$$N = as^{-b}$$

6.2 Variable amplitude loading

Palmgren-Miner rule

$$\sum \frac{n_i}{N_i} = \text{ damage}$$

6.3 Effect of notches

Stress concentration factor

$$K_t = \frac{\text{Maximum stress in notch}}{\text{Nominal stress remote from notch}}$$

Strength reduction factor

$$K_f = \frac{\text{Allowable stress without notch}}{\text{Allowable stress with notch}} \approx \frac{\text{Endurance limit without notch}}{\text{Endurance limit with notch}}$$

notch sensitivity

$$Q = \frac{K_f - 1}{K_t - 1}$$

6.4 Effect of mean load

$$S_a = S_o\left(1 - \left(\frac{S_m}{S_{ut}}\right)^n\right)$$

n = 1 (Goodman line)
n = 2 (Gerber parabola)

7.0 Friction and wear

7.1 Hardness

Defined as

$$H = \frac{\text{Load}}{\text{Projected area of indent}}$$

less commonly as

$$H = \frac{\text{Load}}{\text{Surface area of indent}}$$

For metals

$$H \approx 3s_y$$

7.2 Surface finish parameters

If z is the height of the surface profile measured from a mean line through the profile

7.2.1 R_a is the arithmetic mean of the departures of the profile from the mean line

$$R_a = \frac{1}{L}\int_0^L |z(x)dx|$$

7.2.2 R_q is the rms variation

$$R_q = \sqrt{\frac{1}{L}\int_0^L z^2(x)dx}$$

for a Gaussian surface $R_q = \left(\frac{p}{2}\right)^{1/2} R_a$

for a sinusiod surface $R_q = \frac{p}{2\sqrt{2}} R_a$

7.2.3 $s_m = \Delta q$ is the rms slope

$m(x)$ is the surface slope

$$s_m = \sqrt{\frac{1}{L}\int_0^L \left(m(x) - \overline{m}\right)^2 dx}$$

7.2.4 Skew R_{sk} is a measure of the symmetry of the amplitude distribution

$$R_{sk} = \frac{1}{L\left(R_q\right)^3}\int_0^L \left(z(x) - \bar{z}(x)\right)^3 dx$$

7.2.5 Kurtosis is a measure of the 'spikeness' of the profile

$$R_{ku} = \frac{1}{L\left(R_q\right)^4}\int_0^L \left(z(x) - \bar{z}(x)\right)^4 dx$$

7.3 Plasticity index

$$\Psi = \frac{E^*}{H}\left(s_s K_s\right)^{1/2} \approx \frac{E^*}{H} s_m$$

7.4 Wear coefficient

Volume of material lost per unit distance of sliding, Q

$$Q = \frac{KW}{H} = kW$$

K = dimensionless wear coefficient

8.0 Hydrodynamic lubrication

8.1 Navier-Stokes equation

$$r\frac{D\mathbf{u}}{Dt} = \mathbf{F} - \text{grad}P + h\nabla^2\mathbf{u} + \frac{h}{3}\text{grad.div}\mathbf{u}$$

8.2 Sommerfeld number

$$S = \left(\frac{r}{c}\right)^2 \frac{hN}{P}$$

N = angular speed [rev s^{-1}]
P = load per projected area [N m^{-2}]

8.3 Temperature rise variable

$$\Gamma = \frac{pC_p\Delta T}{P}$$

8.4 Radial flow variable

$$\frac{Q}{rcNl}$$

8.5 Bearing friction coefficient

This definition applies to hydrodynamic and rolling element bearings

$$m_r = \frac{T}{Wr}$$

9.0 Rolling element bearings

9.1 Reliability

$$R = e^{\left(-\left(\frac{\frac{L}{L_{10}}-x_o}{q-x_o}\right)^b\right)} \approx e^{-\left(\frac{L}{L_{10}q}\right)}$$

9.2 Bearing life

$$\frac{L_1}{L_2} = \left(\frac{W_2}{W_1}\right)^a$$

a = 3 (ball bearings)
a = 10/3 (roller bearings)

$$L = \left(\frac{C}{W}\right)^a \qquad \text{[millions]}$$

9.3 Combined radial and axial loads

$$W_e = XVW_r + YW_a$$

$V = 1$ rotating inner ring
$V = 1.2$ rotating outer ring

$$e = f\left(\frac{W_a}{C_o}\right) = g\left(\frac{W_a}{izD^2}\right)$$

If $e \geq \frac{W_a}{VW_r}$ then $X = 1$, $Y = 0$

10.0 Elementary statistics

10.1 General

Mean $\bar{x} = \frac{1}{N}\sum f_i x_i$ where $N = \sum f_i$

Variance

$$s^2 = \frac{1}{N}\sum f_i(x_i - \bar{x})^2 = \frac{1}{N}\sum f_i x_i^2 - \bar{x}^2$$

10.2 Normal distribution

Variance of the mean computed from a normally distributed sample

$$s_{\bar{x}}^2 = \frac{s_x^2}{N}$$

Standard normalized variate $u[0,1]$

$$u = \frac{x - \bar{x}}{s}$$

11.0 Kinematics

11.1 Mobility

Mobility of a three dimensional rigid body mechanism

$$M = 6(n-1) - \sum_{i=1}^{j} c_i$$
$$= 6(n-j-1) + \sum_{i=1}^{j} f_i$$

Planar mechanism (Grubler's formula)

$$M = 3(n-1) - 2j_1 - j_2$$

12.0 Screws and fasteners

12.1 Stress

Tensile stress in a thread

$$s_t = \frac{W}{A_t}$$

$$A_t = \frac{p}{4}\left(\frac{d_P + d_r}{2}\right)^2$$

Stress concentration factor is usually between 3 - 4 in the region where the bolt meets the nut

12.2 Power screws

For a square threaded screw

$$T = \frac{d_m}{2}\frac{F\left(m\cos(g) \pm \sin(g)\right)}{\cos(g) \mp m\sin(g)}$$

12.3 Bolted members

Load on bolt

$$F_i + \frac{F_E k_b}{k_b + k_m}$$

Load between members

$$\frac{F_E k_m}{k_m + k_b} - F_i$$

Stiffness of clamped members
Case 1: Clamped member pair

$$k_m = \frac{pEd}{2\ln\left(5\frac{l + d/2}{l + 5d/2}\right)}$$

Case 2: Single member clamped to rigid body

$$k_m = \frac{pEd}{\ln\left(5\frac{l + d/2}{l + 5d/2}\right)}$$

Torque requirements

$$T = KF_i d$$

$$K = \left(\frac{d_m}{2d}\right)\left(\frac{\tan g + m\sec a}{1 - m\tan g\sec a}\right) + 0.625m_c$$

13.0 Belts

To make belt go around pulley

$$T_o = mv^2$$

m is mass per length of pulley
Condition for avoidance of slip

$$\ln\frac{T_1}{T_2} = m'q$$

$$m' = m\operatorname{cosec}(a/2)$$

Power transmitted on point of slipping

$$T_1\left(1 - e^{-m'q}\right)v$$

14.0 Gears

14.1 Definitions

Number of teeth *N*
Pitch circle diameter *d*
Pressure angle ϕ
Base circle diameter $d_b = d\cos f$
Circumferential pitch $p = pd/N$
Module $m = d/N$

14.2 Meshing gears same module

$$\frac{N_1}{N_2} = \frac{d_1}{d_2} = \frac{w_2}{w_1}$$

14.3 Gearboxes

14.3.1 Compound gearbox

Gear ratio, *R*, between input and output shaft

$$R = \frac{\text{product of teeth on followers}}{\text{product of teeth on drivers}} = \frac{\text{speed of driver}}{\text{speed of follower}}$$

14.3.2 Epicyclic gearboxes

Basic ratio R_o defined as ratio of the gearbox with the planets held stationary

$$R_o = \frac{\text{speed of fastest shaft}}{\text{speed of second shaft}}$$

14.4 Tooth stress

$$s = \frac{F_t}{K_v bmY} \quad \text{(fatigue not important)}$$

$$s = \frac{F_t}{K_v bmJ} \quad \text{(fatigue important)}$$

15.0 Springs

15.1 Helical springs/axial deflection

Shear stress on inside edge of spring

$$t = \frac{8FD}{pd^3} + \frac{4F}{pd^2}$$

$$= K\frac{8FD}{pd^3}$$

Linear stiffness

$$k = \frac{d^4G}{8D^3N}$$

15.2 Helical springs/torsional deflection

Bending stress

$$s = K\frac{32Fr}{pd^3}$$

Angular stiffness

$$k = \frac{Fr}{q} = \frac{d^4E}{64DN}$$